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V. N. Diyesperov and O. S. Ryzhov

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ON BODIES OF ROTATION IN MACH 1 FLOW OF AN IDEAL GAS

V. N. Diyesperov and O. S. Ryzhov

The first solution of the problem of asymptotic modes of damping of perturbations at large distances from a finite body at Mach 1 flow was given in [1, 2], where the gas was assumed to be ideal, that is, nonviscous and thermally nonconducting. Under the assumption of a self-semilar form of the desired integral, those two papers presented a system of ordinary differential equations which describes the entire velocity field of the flow. The derived equations were integrated by numerical methods. In [3], the solution (in its parametric representation) was written in the form of simple algebraic functions. The value of the exponent appearing in the definition of the self-similar variable was also found in [3]. The parametric form of the solution was also indicated in [4]. /164*

In [5, 6], corrections were determined for the asymptotic formulas. The solution of this problem makes it possible not only to calculate more accurately the parameters of the gas, but also to find the relationship between them and the resistance force acting on the body. It is significant when one replaces the body around which the flow is taking place with a dipole, the resistance force must become zero. Below, we shall give a solution for the velocity field of a source and enabling us to calculate the effect of that force. It turns out that the source and the dipole perburb a uniform Mach 1 flow ahead of a shock front to the same extent. The parameters of the flows that arise due to the effect of these phenomena differ only in the region behind shock.

We shall assume that far away from the body of rotation, the motion of the ideal gas (as defined above) is isentropic. In fact, the flow is intersected by a shock wave [2, 3], but its intensity is small, and the change caused by it in the entropy is very much smaller than the quantities that are retained in the approximation to be considered below. Since a flow originating at infinity is uniform, it will (under the assumption made) be irrotational everywhere. We can then transform from a system of Euler equation to a single partial differential equation for /165
the velocity potential.

Let x and r denote cylindrical coordinates, let v_x and v_r denote the projections of the velocity vector along these axes, let φ denote the potential, let a denote the speed of sound and let w be the specific enthaply. As we know [7],

$$\left[a^2-\left(\frac{\partial\varphi}{\partial x}\right)^2\right]\frac{\partial^2\varphi}{\partial x^2}-2\frac{\partial\varphi}{\partial x}\frac{\partial\varphi}{\partial r}\frac{\partial^2\varphi}{\partial x\,\partial r}+\left[a^2-\left(\frac{\partial\varphi}{\partial r}\right)^2\right]\frac{\partial^2\varphi}{\partial r^2}+\frac{a^2}{r}\frac{\partial\varphi}{\partial r}=0, \quad (1)$$

*Numbers in the margin represent pagination in the foreign text.

$$\tfrac{1}{2}(v_x^2+v_r^2)+w=\tfrac{1}{2}a_*^2+w_*, \qquad v_x=\frac{\partial\varphi}{\partial x}, \qquad v_r=\frac{\partial\varphi}{\partial r}. \tag{2}$$

Henceforth, an asterisk refers to the parameters of a gas in a critical state. To determine, from the Bernoulli integral (2), the speed of sound in terms of the derivatives of the potential with respect to the coordinates, we need to use the equation of state of the medium; it gives the pressure as a function of the specific volume V (or density $\rho = 1/V$) and the specific entropy s. Increase in the entropy of the front of a weak shock wave is proportional to the cube of the wave amplitude [7], a quantity conveniently represented by the difference in the specific enthalpies on the two sides of the surface of discontinuity. Henceforth, we can confine ourselves to the approximation

$$a=a_*+\left(\frac{\partial a}{\partial w_*}\right)_s(w-w_*)+\ldots. \tag{3}$$

In accordance with the first law of thermodynamics, in an adiabatic process the increase in the specific enthalpy dw = Vdp, from which follows that

$$\left(\frac{\partial a}{\partial w}\right)_s=\left(\frac{\partial a}{\partial \rho}\right)_s\left(\frac{\partial \rho}{\partial p}\right)_s\left(\frac{\partial p}{\partial w}\right)_s=\frac{m-1}{a}, \qquad m=\frac{1}{2\rho^3a^2}\left(\frac{\partial^2 p}{\partial V^2}\right)_s. \tag{4}$$

Formula (4) enables us to express the coefficient in the expansion (3) for the speed of sound in terms of a derivative describing the Poisson adiabate for a moving gas. By using Bernoulli's integral, we can now transform Eq. (1) to a single velocity potential φ. We note that, for a perfect gas with ratio of specific heat capacities $\varkappa$, the relation between the speed of sound and the specific enthalpy is given by the simple formula $a=\sqrt{[(\varkappa-1)w]}$, in which the entropy does not occur.

Let us suppose that the velocity of the perturbed flow is equal in magnitude to the critical velocity a_* and is directed along the x-axis. Let us seek the solution of Eq. (1) in the form of the expansion

$$\varphi=a_*\left[x+\sum_i\varphi_{\omega_i}(x,r)\right], \qquad \varphi_{\omega_i}(x,r)=r^{\omega_i}f_{\omega_i}(\xi), \qquad \xi=\frac{x}{(2m_*)^{1/3}r^n}. \tag{5}$$ /166

The function φ_{ω_0} is an integral of Kármán's approximate equation [8] and it gives the asymptotic modes of damping of perturbations far away from a body streamlined by a uniform flow which is Mach 1 at infinity [1, 2]. As shown in

[3], the parameter n of expansion (5) is equal to 4/7, and the first power is $\omega_0 = -{}^2/_7$. The function $f_{-2/7}$ satisfies the ordinary differential equation

$$\left(\frac{df_{-2/7}}{d\xi} - \frac{16}{49}\xi^2\right)\frac{d^2 f_{-2/7}}{d\xi^2} - \frac{32}{49}\xi\frac{df_{-2/7}}{d\xi} - \frac{4}{49}f_{-2/7} = 0. \tag{6}$$

For the exact solution of this equation in a mixed subsonic and supersonic region situated in front of the shock front, it is convenient to use the parametric formulas

$$\begin{aligned} \xi &= b_1\sigma^{-5/7}(1-\sigma)^{-2/7}, \\ f_{-2/7} &= -\frac{1}{9}b_1^3\sigma^{-15/7}(1-\sigma)^{-6/7}(2-30\sigma^2+28\sigma^3). \end{aligned} \tag{7}$$

developed in [3]. The choice of the constant b_1 remains arbitrary. In the region behind the shock front, we may take, in analogy with (7),

$$\begin{aligned} \xi &= b_2\sigma^{-5/7}(1+\sigma)^{-2/7}, \\ f_{-2/7} &= \frac{1}{9}b_2^3\sigma^{-15/7}(1+\sigma)^{-6/7}(2-30\sigma^2-28\sigma^3). \end{aligned} \tag{8}$$

The value of the constant b_2 depends on the value of the constant b_1 and is found by piecing together the integrals (7) and (8) on the shock wave front.

Let us estimate first of all the accuracy of the approximation used. In formula (3) for the speed of sound, the discarded terms are of the order of $r^{-12/7}$. They must be taken into account in the derivation of the recursion equations from which one finds functions φ_{ω_i} and f_{ω_i} with $\omega_i \leqslant -{}^8/_7$, but, in our case, these functions are of no interest.

Suppose now that we denote by $\mathbf{v} = \operatorname{grad}\varphi$ the vector of velocity of the gas particles and by T the gas temperature. In accordance with Crocco's equation,

$$\mathbf{v} \times \mathbf{curl} = T \operatorname{grad} s,$$

since, in flows that equalize out at infinity, Bernoulli's constant does not change on shifting from one streamline to another [7]. If the motion of the gas has axial symmetry, the vector curl v is orthogonal to the velocity vector v. Therefore,

$$|\,\mathrm{curl}\,\mathbf{v}| = \frac{T_*}{a_*}|\mathrm{grad}\,s| + \ldots .$$

On crossing the shock wave front, the change in pressure is of the same order as the change in the absolute value v of the particle velocity, that is, it is proportional to $r^{-6/7}$. The increase in the entropy is proportional to $r^{-18/7}$. At a great distance from the streamlined body, the form of the shock wave is given by the asymptotic equation $x = \xi_2 r^{4/7}$, where $\xi_2 = \mathrm{const}$. The entropy gradient, and with it $|\mathrm{curl}\,\mathbf{v}|$, approaches zero at the same rate as $r^{-25/7}$ when $r \to \infty$. This estimate asserts that functions φ_{ω_i} and f_{ω_i} with $\omega_i > -{}^8/_7$ may be determined when we assume that the flow is irrotational. /167

Let us denote by a subscript 1 the parameters of the gas on that side of the shock wave surface that is directed toward the oncoming flow and let us denote by a subscript 2 the values of the parameters of the medium on the opposite side. By v_n and V_τ we denote the components of the velocity vector in the projections onto the normal and tangent to the shock front respectively. Let us consider Hugoniot's conditions which must be satisfied on crossing the shock wave. As we know [7], there are four of these and, in the approximation that we are making, two conditions are automatically satisfied. Specifically, in accordance with the first of these, the pressure and density are adiabatically related to an accuracy of terms of second order of smallness:

$$p_2 - p_1 = \left(\frac{\partial p}{\partial \rho_1}\right)_s (\rho_2 - \rho_1) + \frac{1}{2}\left(\frac{\partial^2 p}{\partial \rho_1^2}\right)_s (\rho_2 - \rho_1)^2 + \ldots . \tag{9}$$

The second of these conditions is that the pressure behind the shock front be expressed in terms of the pressure ahead of it with the aid of Bernoulli's integral, which holds also for the description of discontinuous motion. Furthermore, the product of the two normal components of the velocity vector of gas particles obeys the condition [7]

$$v_{n2} v_{n1} = \frac{p_2 - p_1}{\rho_2 - \rho_1}.$$

By using the expansion (9) and making the specific enthalpy w, the independent thermodynamic variable, we have

$$v_{n2} v_{n1} = a_*^2 + (m_* - 1)(w_2 + w_1 - 2w_*) + \ldots . \tag{10}$$

Finally, the last of the Hugoniot conditions simply reduces to the requirement that the tangential component of the velocity vector remain continuous on crossing the shock front. Obviously, this can be replaced by the condition that the potential itself be continuous. We have

$$\varphi_2 = \varphi_1. \tag{11}$$

The omitted terms in (10) affect only the determination of the functions φ_{ω_i} and f_{ω_i} with $\omega_i \leqslant -8/7$. We note that inclusion of the square terms in formula (10) is impossible in the framework of adiabatic approximation.

In the region upstream from the shock front, the power ω_1 following ω_0 in the expansion (5) is -6/7, according to [5, 6]. Let us show that this solution corresponds to replacement, in the first approximation, of the streamlined body by a dipole, if we use it also in the region behind the shock wave. To do this, let us construct a control surface around the body and let us calculate the quantity /168
Q of gas that passes through this surface. The control surface shall be a cylindrical surface of radius r whose generators are parallel to the x-axis. Then,

$$Q = 2\pi r \int_{-\infty}^{+\infty} \rho v_r \, dx. \tag{12}$$

Turning to the expansion (5), we easily find

$$Q = 2\pi\rho_* a_* r \int_{-\infty}^{+\infty} \left(\frac{\partial \varphi_{-5/7}}{\partial r} + \frac{\partial \varphi_{-6/7}}{\partial r} \right) dx + \ldots,$$

$$\frac{\partial \varphi_{-5/7}}{\partial r} = -\frac{2}{7} r^{-9/7} \left(f_{-5/7} + 2\xi \frac{df_{-5/7}}{d\xi} \right), \qquad \frac{\partial \varphi_{-6/7}}{\partial r} = -\frac{2}{7} r^{-13/7} \left(3f_{-6/7} + 2\xi \frac{df_{-6/7}}{d\xi} \right). \tag{13}$$

Let us look at the first of the integrals in (13). To get an estimate for it, we first of all take into account the fact that the equation of the shock front is simply $\xi = \xi_2$.* We have

$$r \int_{-\infty}^{+\infty} \frac{\partial \varphi_{-5/7}}{\partial r} \partial x = -\frac{2\,(2m_*)^{1/4}}{7} r^{2/7} \int_{-\infty}^{+\infty} \left(f_{-5/7} + 2\xi \frac{df_{-5/7}}{d\xi} \right) d\xi = -\frac{2\,(2m_*)^{1/4}}{7} r^{2/7} I.$$

If we now let the radius of the control cylindrical surface approach infinity, the expression that we have written will also approach infinity at the same rate as $r^{2/7}$ under the condition that $I \neq 0$.

The realization of such a situation would indicate that the streamlined body introduces perturbations into the oncoming flow, and that the perturbations result in infinite gas flow through any surface surrounding the flow. Of course, the perturbations from a finite body cannot be this strong. Therefore, it is necessary that I = 0.

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*Later, we shall make this estimate more precise.

To check this assertion, let us partition the integral I into two parts:

$$I = I_1 + I_2 = \left(\int\limits_{-\infty}^{\xi_2} + \int\limits_{\xi_2}^{+\infty} \right) \left(f_{-2/7} \text{ [illegible] } \frac{df_{-2/7}}{d\xi} \right) d\xi.$$

Here, the coordinate of the shock wave $\xi =$ [illegible] is calculated in accordance with the remark made above regarding the first approximation [2, 3]. In formulas (7), let us shift to the new parameter $\eta = {}^5/_{12}(1 - \sigma^{-1})$ and let us set $b_1 = 5^{5/7} 12^{2/7} 7^{-1}$.

Then, the limiting characteristics of the flow in the first approximation will correspond to $\xi = \eta = 1$. A new parametric representation of the function $f_{-2/7}$ in the region situated in front of the shock front is written

$$\xi = \frac{12\eta - 5}{7\eta^{5/7}}, \quad f_{-2/7} = 2^5 7^{-3} \eta^{2/7} (12\eta^2 - 15\eta - 25). \tag{14}$$

For the region behind the shock wave, a different substitution, namely, $\zeta = -{}^5/_{12}(1 + \sigma^{-1})$, is more convenient. As a result of this substitution, we have /169

$$\xi = b_3^{2/7} \frac{12\zeta + 5}{7\zeta^{5/7}}, \qquad f_{-2/7} = 2^5 7^{-3} b_3^{4/7} \zeta^{2/7} (12\zeta^2 + 15\zeta - 25),$$
$$b_3^{2/7} = -5^{-5/7}\, 7 \cdot 12^{-2/7} b_2. \tag{15}$$

Let us satisfy conditions (10) and (11), which must hold upon passage through the shock wave. The requirement of continuity of the potential with $\xi = \xi_2$ leads to the equation $f_{-2/7,\,2} = f_{-2/7,\,1}$, whereas the relationship between the normal components of the velocity vector yields

$$\frac{df_{-2/7,\,2}}{d\xi} + \frac{df_{-2/7,\,1}}{d\xi} = \frac{32}{49} \xi_2^2. \tag{16}$$

From this we can easily obtain the values of the parameters $\eta = \eta_2$ and $\zeta = \zeta_2$, corresponding to the shock wave front. In accordance with [6], we have

$$\eta_2 = 12^{-1}(7\sqrt{3} + 12), \quad \zeta_2 = 12^{-1}(7\sqrt{3} - 12), \quad \xi_2 = 2 \cdot 3^{1/7} (\sqrt{3} - 1)^{1/7}.$$

At the same time, we determine the quantities

$$b_3 = (2 - \sqrt{3})^{1/3}, \qquad b_2 = 5^{5/7} 7^{-1} 12^{2/7} (\sqrt{3} - 2)^{1/7}.$$

Thus, as ξ ranges from $-\infty$ to ξ_2, the parameter η increases monotonically from 0 to η_2. As ξ ranges from ξ_2 to $+\infty$, the parameter ζ decreases monotonically from ζ_2 to 0. Let us now substitute the representation (14) of the function $f_{-2/7}$

into the expression for the integral I_1. When we do this, we obtain

$$I_1 = \frac{2^6 15}{7^4} \int_0^{\eta_2} (24\eta^{13/7} - 26\eta^{6/7} - 5\eta^{-1/7})\, d\eta = -2^{23/7} 3^{4/7} 7^{-2} 13^{-1} (7 - 4\sqrt{3})^{1/7}.$$

An analogous transformation of the integral I_2 based on formulas (15) leads to the value

$$I_2 = -\frac{2^6 15}{7^4} b_3^{12/7} \int_0^{\zeta_2} (24\zeta^{13/7} + 26\zeta^{6/7} - 5\zeta^{-1/7})\, d\zeta =$$
$$= 2^{23/7} 3^{4/7} 7^{-2} 13^{-1} (7 + 4\sqrt{3})^{1/7} (7 - 4\sqrt{3})^{2/7}.$$

As one can easily see, the two values obtained are equal in magnitude but opposite in sign. It follows from this that I = 0. In accordance with the calculations that we have made, we can obtain from (13) the estimate $Q \sim r^{-2/7}$. In this estimate, we take into consideration the fact that the form of shock wave differs from that of the curve $\xi = \xi_2$. By letting the radius of the control cylindrical surface become infinite, we conclude that Q = 0. If the number ω_1 following ω_0 in the expansion (5) is equal to -6/7, the flow around a finite body takes place according to the scheme corresponding to a perturbation of the original uniform flow by a dipole. Specifically, as is well known [7], behind the streamlined body there forms an eddy wake due to the presence of dissipative processes in an actual medium. If we assume that the gas is nonviscous and thermally nonconducting, we can calculate this accompanying wake by adding the singularities introduced by the source to in the flow scheme. In such a scheme, obviously, $Q \neq 0$. As is easily shown, one can satisfy the above-mentioned requirement by setting $\omega_1 = -4/7$, and $\omega_2 = -6/7$ in the expansion (5). It follows from [5, 6] that the function $\varphi_{-4/7}$ must be identically equal to zero in the region in front of the shock wave because otherwise the flow velocity field would have singularities on its limiting characteristics. Consequently, to obtain a singularity of the source type, we need to take $\varphi_{-4/7} \not\equiv 0$ in the region behind the front of the shock wave. Attention was first paid to such a possibility in [9] in a study of plane-parallel transonic flows. /170

Substitution of the expansion (5) into the original equations of motion (1) leads to a linear differential equation for the function $f_{-4/7}$. This equation is homogeneous:

$$\left(\frac{df_{-3/7}}{d\xi} - \frac{16}{49}\xi^2\right)\frac{d^2 f_{-4/7}}{d\xi^2} + \left(\frac{d^2 f_{-3/7}}{d\xi^2} - \frac{48}{49}\xi\right)\frac{df_{-4/7}}{d\xi} - \frac{16}{49} f_{-4/7} = 0. \tag{17}$$

The differential equation for the function $f_{-6/7}$ is found to be nonhomogeneous:

$$\left(\frac{df_{-2/7}}{d\xi}-\frac{16}{49}\xi^2\right)\frac{d^2f_{-6/7}}{d\xi^2}+\left(\frac{d^2f_{-2/7}}{d\xi^2}-\frac{64}{49}\xi\right)\frac{df_{-6/7}}{d\xi}-\frac{36}{49}f_{-6/7}=-\frac{df_{-4/7}}{d\xi}\frac{d^2f_{-4/7}}{d\xi^2}. \qquad (18)$$

To make more precise the description of the velocity field for purpose of determination of the form of the shock wave, we can no longer confine ourselves to the simple equation $\xi = \xi_2$. We take the equation of the front in the form

$$\xi = \xi_2(1 + c_1r^{-2/7} + c_2r^{-4/7} + \ldots) = \xi_2(1+\Delta). \qquad (19)$$

When we satisfy, the condition of continuity of the potential on crossing of the shock wave, we obtain first the equation

$$f_{-6/7,\,2} = c_1\xi_2\left(\frac{df_{-2/7,\,1}}{d\xi}-\frac{df_{-2/7,\,2}}{d\xi}\right), \qquad (20)$$

and then the equation

$$f_{-6/7,\,2}-f_{-6/7,\,1} = -\xi_2\left[c_2\left(\frac{df_{-2/7,\,2}}{d\xi}-\frac{df_{-2/7,\,1}}{d\xi}\right)+\frac{1}{2}c_1^2\xi_2\left(\frac{d^2f_{-2/7,\,2}}{d\xi^2}-\frac{d^2f_{-2/7,\,1}}{d\xi^2}\right)+c_1\frac{df_{-4/7,\,2}}{d\xi}\right], \qquad (21)$$

which the values of the function $f_{-6/7}$ must obey on both sides of the surface of discontinuity. Let us look at the unit normal n to the shock front, with components n_x and n_r along the x- and r-axes, respectively. To calculate the component $v_n = v_xn_x + v_rn_r$ of the velocity vector, we need, in the first approximation, to know only /171

$$n_x = 1 - {}^{8}/_{49}(2m_*)^{2/3}\xi_2^2(r^{-6/7} + c_1r^{-8/7} + {}^{1}/_{4}c_1^2r^{-10/7} + \ldots).$$

Turning to the boundary condition (10), we easily derive the equation

$$\frac{df_{-4/7,\,2}}{d\xi} = c_1\xi_2\left[\frac{32}{49}\xi_2-\left(\frac{d^2f_{-2/7,\,2}}{d\xi^2}+\frac{d^2f_{-2/7,\,1}}{d\xi^2}\right)\right]. \qquad (22)$$

On the other hand, the relationship between the first derivatives of the function $f_{-6/7}$ on the two sides of the surface of discontinuity takes the form

$$\frac{df_{-4/7,\,2}}{d\xi}+\frac{df_{-4/7,\,1}}{d\xi}=\xi_2\left[\frac{8}{49}c_1{}^2\xi_2-c_2\left(\frac{d^2f_{-2/7,\,2}}{d\xi^2}+\frac{d^2f_{-2/7,\,1}}{d\xi^2}\right)-\right.$$
$$\left.-c_1\frac{d^2f_{-4/7,\,2}}{d\xi^2}-\frac{1}{2}c_1{}^2\xi_2\left(\frac{d^3f_{-2/7,\,2}}{d\xi^3}+\frac{d^3f_{-2/7,\,1}}{d\xi^3}\right)\right]. \tag{23}$$

We shall now deal with the transformation of the equations (20) and (22), which give the values of the function $f_{-4/7}$ and its first derivative on the shock wave. If we eliminate the constant c_1 from them, we get

$$\left(\frac{df_{-2/7,\,2}}{d\xi}-\frac{df_{-2/7,\,1}}{d\xi}\right)\frac{df_{-4/7,\,2}}{d\xi}+\left[\frac{32}{49}\xi_2-\left(\frac{d^2f_{-2/7,\,2}}{d\xi^2}+\frac{d^2f_{-2/7,\,1}}{d\xi^2}\right)\right]f_{-4/7,\,2}=0.$$

From this differential equation, we now eliminate the second derivatives of the function $f_{-2/7}$ by means of Eq. (6), which applies to both sides of the surface of discontinuity. Keeping condition (16) in mind, we find

$$\frac{d^2f_{-2/7,\,2}}{d\xi^2}+\frac{d^2f_{-2/7,\,1}}{d\xi^2}=\frac{64}{49}\xi_2,$$

after which it is easy to obtain the desired formula

$$\left(\frac{df_{-2/7,\,2}}{d\xi}-\frac{16}{49}\xi_2^2\right)\frac{df_{-4/7,\,2}}{d\xi}-\frac{16}{49}\xi_2 f_{-4/7,\,2}=0. \tag{24}$$

Let us now follow by integrating equation (17), rewriting it in the following form:

$$\frac{d}{d\xi}\left[\left(\frac{df_{-2/7}}{d\xi}-\frac{16}{49}\xi^2\right)\frac{df_{-4/7}}{d\xi}-\frac{16}{49}\xi f_{-4/7}\right]=0.$$

It follows from (24) that the constant of integration must be 0. A second integration yields

$$f_{-4/7}=2^{-40/49}3^{-13/49}7^{8/7}(7-4\sqrt{3})^{2/7}A\exp\left(16\int_{\xi_2}^{\xi}\xi\left[49\frac{df_{-2/7}}{d\xi}-16\xi^2\right]^{-1}d\xi\right), \tag{25}$$

where the constant A is proportional to the coefficient c_1 in the expansion (19) /172
for the wave front:

$$A = 2^{109/35} 3^{19/70} 5^2 7^{-12/5} (11\sqrt{3} - 19)^{-1/7} c_1.$$

Let us now establish a relationship between the constant A and the gas flow Q. When we substitute the expansion (5) into formula (12), we see immediately that integration of the function $r\partial\varphi_{-4/7}/\partial r$ yields a finite value independent of the radius of the cylindrical control surface. For $\varphi_{-4/7} \neq 0$ the integral I in Eq. (13) is of the order $r^{-2/7}$. The product $r^{2/7}I$ remains constant as $r \to \infty$ and $I \to 0$, On the other hand, if $\varphi^{-4/7} = 0$, then $I \sim r^{-4/7}$ and $r^{2/7}I \sim r^{-2/7} \to 0$ as we increase without bound the radius of the control surface. The estimates we have given do not contradict the calculations which were made earlier and according to which the integral I vanishes identically. The point is that, when we make the description of the velocity field of the flow more precise, the form of the shock wave must be given by the more precise relation (19) rather than by the simple equation $\xi = \xi_2$. As a result, the gas flow is

$$Q = -\frac{8}{7}\pi\rho_* a_* (2m_*)^{1/2} \left[r^{2/7} \int\limits_{\xi_2}^{\xi_2(1+\Delta)} 2\xi_2 \left(\frac{df_{-2/7,1}}{d\xi} - \frac{df_{-2/7,2}}{d\xi} \right) d\xi + \right.$$
$$\left. + \int\limits_{\xi_2}^{+\infty} \left(f_{-4/7} + \xi \frac{df_{-4/7}}{d\xi} \right) d\xi \right].$$

By using the boundary conditions (20), let us transform this expression to the form

$$Q = -\frac{8}{7}\pi\rho_* a_* (2m_*)^{1/2} \left[2\xi_2 f_{-4/7,2} + \int\limits_{\xi_2}^{+\infty} \left(f_{-4/7} + \xi \frac{df_{-4/7}}{d\xi} \right) d\xi \right],$$

which is entirely determined by the values of the function $f_{-4/7}$ and its first derivative. Also, we find

$$Q = -\frac{8}{49}\pi\rho_* a_* (2m_*)^{1/2} b_3^{8/7} [2^{-4/3} 3^{-1/6} 7^{7/6} (\sqrt{3} - 1) + 5] A.$$

Let us now see about integrating (18). One can easily see that the desired solution $f^{01}_{-4/7}$ of the homogeneous equation corresponding to it is

$$f^{01}_{-4/7} = B \frac{df_{-2/7}}{d\xi}, \qquad B = \text{const.} \tag{26}$$

To see this, let us recall Eq. (6), which the function $f_{-2/7}$ must satisfy. Differentiating it with respect to ξ, we see that Eq. (26) is valid. As was mentioned [5], we may, without loss of generality, take the constant B = 0 in the region upstream from the wave front. This is true because the original partial differential equation (1) is invariant with respect to shift along the x-axis. Keeping this fact in mind, let us take the basic singularity in the expansion (5) not at the coordinate origin but at some point $x = x_0$ on the $(r = 0)$-axis. Then, /173

$$\varphi_{-2/7}(x + x_0, r) = \varphi_{-2/7}(x, r) + \frac{\partial \varphi_{-2/7}(x, r)}{\partial x} x_0 + \ldots$$

$$\ldots = r^{-2/7} f_{-2/7}(\xi) + (2m_*)^{-1/3} r^{-6/7} \frac{df_{-2/7}}{d\xi} x_0 + \ldots.$$

Comparison of (26) with this equation shows that

$$B = (2m_*)^{-1/3} x_0.$$

If $x_0 = 0$, then the constant B is also 0. The function $f_{-4/7}$, whose derivatives determine the right-hand member of Eq. (18), is nonzero only in the region behind the shock wave. For $B = 0$, the function $f_{-6/7}$ is also nonzero in that region but it vanishes identically upstream from the shock front. Under condition (21), the quantity $f_{-6/7,\,1} = 0$. Also, under condition (23), the derivative $df_{-6/7,\,1} / d\xi = 0$.

To obtain a solution of the nonhomogeneous Eq. (18), it is convenient to use as the independent variable in all the transformations the parameter ξ, rather than 1. When we substitute the representation (15) of the function $f_{-2/7}$ into formula (26) and set $B = 2^{-4} 7^2 b_3^{-4/7}$ for convenience, we obtain

$$f^{01}_{-6/7} = \zeta^{3/7}(6\zeta + 5). \tag{27}$$

In accordance with Liouville's formula, we find a second linearly independent solution of the homogeneous equation corresponding to (18). Let us adopt the notation $f^{02}_{-6/7}$ for it. Then, simple calculations yield

$$f^{02}_{-6/7} = \zeta^{3/7}(6\zeta + 5) \int\limits_{\zeta_2}^{\zeta} \frac{d\zeta}{\zeta(\zeta + 1)^{1/5}(6\zeta + 5)^2}. \tag{28}$$

The transformation to the independent variable ζ in the integral (25) enables us to evaluate it in closed form:

$$f_{-4/7} = A\zeta^{2/7}(\zeta + 1)^{2/5}.$$

The right-hand member of equation (18) is

$$h_{-6/7} = \frac{df_{-4/7}}{d\xi} \frac{d^2 f_{-4/7}}{d\xi^2} = \frac{7^3}{2 \cdot 3^2 5^5} A^2 b_3^{-4/7} \frac{\zeta^{10/7}}{(\zeta + 1)^{1/5}(6\zeta - 1)^4} [(12\zeta + 5)(72\zeta^3 + 48\zeta^2 + 1105\zeta + 100)].$$

Now, it is easy to write the desired solution of the nonhomogeneous equation (18). Let us denote by

$$W = \frac{49}{10} b_3^{-1/7} \zeta^{6/7} (\zeta + 1)^{-1/6} (6\zeta - 1)^{-1}$$

the Wronskian of the linearly independent integrals (27) and (28). For the region downstream from the shock front, we obtain

$$f_{-6/7} = C f_{-6/7}^{01} + f_{-6/7}^{01}(\xi) \int_{\xi_s}^{\xi} \frac{h_{-6/7}(\xi) f_{-6/7}^{02}(\xi)\, d\xi}{U(\xi) W(\xi)} - f_{-6/7}^{02}(\xi) \int_{\xi_s}^{\xi} \frac{h_{-6/7}(\xi) f_{-6/7}^{01}(\xi)\, d\xi}{U(\xi) W(\xi)}. \tag{29}$$

When we finally transform to the parameter ζ in both integrals, we need to know the expression for the difference /174

$$U = \frac{df_{-4/7}}{d\xi} - \frac{16}{49} \xi^2 = \frac{2^4 5^2}{7^4} b_3^{6/7} \zeta^{-4/7} (\zeta + 1)(6\zeta - 1).$$

It remains to satisfy conditions (21) and (23) on the front of the shock wave. First of all, we note that, for $f_{-4/7} = 0$, the function $f_{-6/7}$ vanishes identically not only in the region in front of the shock wave, but also in the region behind it. In other words, the representation (26) automatically takes care of the boundary conditions on the wave front. Let us now eliminate from (21) and (23) the constant c_2. We substitute into the relationship so obtained the integral (29). This yields

$$C = -2^{-281/35} 3^{-133/70} 5^{-4} 7^{14/5} (97 - 56\sqrt{3})^{1/7} (1901\sqrt{3} - 2413) A^2.$$

These last expressions lead to the equation

$$c_2 = 2^{481/35} 3^{-283/70} 5^{-6} 7^{24/5} (3\sqrt{3} - 5)^{1/7} (77\,709\sqrt{3} - 136\,669) A^2.$$

Let us investigate the asymptotic behavior of the components of the velocity vector close to the axis of symmetry. By letting r approach zero in our solution, we have, for $x < 0$

$$v_x - a_* = d_1 |x|^{-3/2} + \ldots, \quad v_r = d_2 |x|^{-4} r + \ldots, \tag{30}$$

and, analogously, for $x > 0$

$$\begin{aligned} v_x - a_* &= d_3 x^{-3/2} + d_4 x^{-2} + d_5 x^{-5/2} + \ldots, \\ v_r &= d_6 x^{-4} r + d_7 x^{-9/2} r + d_8 x^{-5} r + \ldots; \end{aligned} \tag{31}$$

Here, the constants $d_1, d_2, \ldots, d_8$ are easily determined from the numerical values of the parameters already used but, because of their cumbersomeness, we shall not write our their explicit expressions. The expansion (30) corresponds to the basic singularity of $\varphi_{-4/3}$. The discarded terms in it are of a higher order of smallness than those left in (31) since $\varphi_{-4/3} = \varphi_{-4/3} = 0$. in the region in front of the front of the shock wave. The constants d_5 and d_8 are proprotional to the coefficient C in the solution (29). With regard to the integral terms in the solution shown, their contribution to the perturbation of the velocity field as $r \to 0$ is, for $x > 0$,

$$v_x - a_* \sim r^2 x^{-6}, \quad v_r \sim r x^{-5}.$$

In conclusion, let us look briefly at the relationships between the integral that we have found for Eq. (1) and the resistance force F_x acting on the body. As we know [7], a portion F'_x of that force is associated with the loss of momentum in the eddy wake of the body. Another portion F''_x, called the wave resistance, can be found by calculation of the x-component of the momentum of the gas transported by the perturbations (per unit time) through the cylindrical control surface. Obviously, $F_x = F_x' + F_x''$. The principal contribution in F''_x is given by the integration of the basic singularity $\varphi_{-4/3}$, from which we get a simple estimate as $r \to \infty$: /175

$$F_x'' = -2\pi\rho_* a_*^2 r \int_{-\infty}^{+\infty} \left(\frac{\partial\varphi_{-4/3}}{\partial x} - 1\right) \frac{\partial\varphi_{-4/3}}{\partial r}\, dx + \ldots \sim r^{-4/3} \to 0.$$

Thus, there will be no wave resistance when the flow around the body has the critical velocity. There will be wave resistance only at strictly supersonic velocities. This very result was established recently in [10], where dissipative processes were taken into account in a real gas. The origin of a resistive force at the critical velocity is explained by the same reason as in an incompressible fluid, namely, the fact that it is due to the displacement of the x-component of the momentum from the wake accompanying beyond the body. It is true that the flow can become eddy not only as a result of intensive friction of the gas in the boundary layer and disruptions behind the object being studied, but also as a result of formation of shock waves (that are by no means weak) close to the body. The process of heat transfer can also have an influence on the formation of a wake in a compressible gas.

Displacement of momentum of a gas is always accompanied by a "deficit" in the flow streamlining the body and flowing through a section of the accompanying wake [7, 10]. If we consider a medium that is nonviscous and thermally nonconducting, the effect of this deficit on the external flow will apparently be equivalent to the effect due to a source of definite intensity. It follows from this that it is impossible to consider the resistance force without including the term with $\varphi_{-4/3}$ in the expansion (5). When $\varphi_{-4/3} = 0$, there will be a flow around a finite body and this flow does not encounter any resistance. In other words, from a purely formal point of view, d'Alembert's paradox remains even if the velocity is Mach 1 at infinity. In this last case, the scheme of the flow corresponds to the interaction of a dipole with an originally uniform flow.

In the asymptotic representation that we have derived for the damping of perturbations at great distances from a finite body, there is a one feature which is identical with the phenomenon observed during operation of a Laval nozzle in an undefined mode. The presence of limiting characteristics and shock waves resulting from them does not enable us to influence the mixed sub- and supersonic flow ahead of the shock front by changing the conditions behind it. Therefore, the principal term in our solution is given by the same function $\varphi_{-2/7}$, regardless of whether it applies to a source or to a dipole. Therefore, the formation of a wake leads only to a small change in the shape of the shock wave and the flow parameters behind it. The constant appearing in the determination of the function $\varphi_{-2/7}$ is in no way associated with the expenditure of the gas flow Q.

REFERENCES /176

1. Guderley, G. and H. Yoshihara. An axial-symmetric transonic flow pattern, Quart. Appl. Math., Vol. 8, No. 2, pp. 333-339, 1951.
2. Barish, D. T. and K. G. Guderley. Asymptotic forms of shock waves in flows over symmetrical bodies at Mach 1, J. Aeronaut. Sci., Vol. 20, No. 7, pp. 491-499, 1953.
3. Fal'kovich, S. V. and I. A. Chernov. Obtekaniye tela vrashcheniya zvukovym potokom gaza (Mach 1 flow of a gas around a body of rotation), Prikl. matem. i mekhan. Vol. 28, No. 2, pp. 280-284, 1964.
4. Müller, E. A. and K. Matchat. Ähnlichkeitslösungen der transsonischen Gleichungen bei der Angström-Machzahl 1, (Similtude solutions for transonic equations at Mach 1), Proc. 11-th Internat. Cong. of Appl. Mechan. Munich, pp. 1061-1068, 1964.
5. Guderley, K. G. Theory of Transonic Flow, Addison-Wesley, 1962.
6. Euvrard, D. Écoulement transsonique à grande distance d'un corps de révolution (Transonic flow at great distances from a body of revolution), Compt. Rendus Acad. Sci., Vol. 260, No. 22, pp. 5691-5692, 1965.
7. Landau, L. D. and Ye. M. Lifshits. Mekhanika sploshnykh sred (Mechanics of continuous media), Moscow, Gostekhizdat, 1954.
8. Karman, Th. The similarity law of transonic flow, J. Math. and Phys., Vol. 26, No. 3, pp. 182-190, 1947.
9. Euvrard, D. Interprétation de la résistance aérodynamique d'un profil d'aile transsonique a l'aide de l'écoulement asymptotique à grande distance, en aval des chocs (Interpretation of aerodynamic resistance of a wing profile in transonic flow, assuming asymptotic flow at a large distance downstream of the shock), Compt. Rendus Acad. Sci., Vol. A 262, No. 7, pp. 405-408, 1966.
10. Ryzhov, O. S. and Ye. D. Trent'yev. O vozmushcheniyakh, kotoryye svyazany sozdaniyem pod"yemnoysily, deystvuyushchey na telo v transzvukovom potoke dissipriruyushchego gaza (On perturbation associated with the appearance of a volumetric force acting on a body in transonic flow of a dissipating gas), Prikl. matem. i mekhan, Vol. 31, No. 6, pp. 1035-1049, 1967.